NYU Yield Curve Seminar - An Overview of Yield Curve Calibration & LIBOR Reform

Nicholas Burgess

nburgessx@gmail.com

6th April 2021

Executive Summary

Yield Curves

- What is a Yield Curve?
- Types of Curves?
- Instruments & Behaviour

Calibration

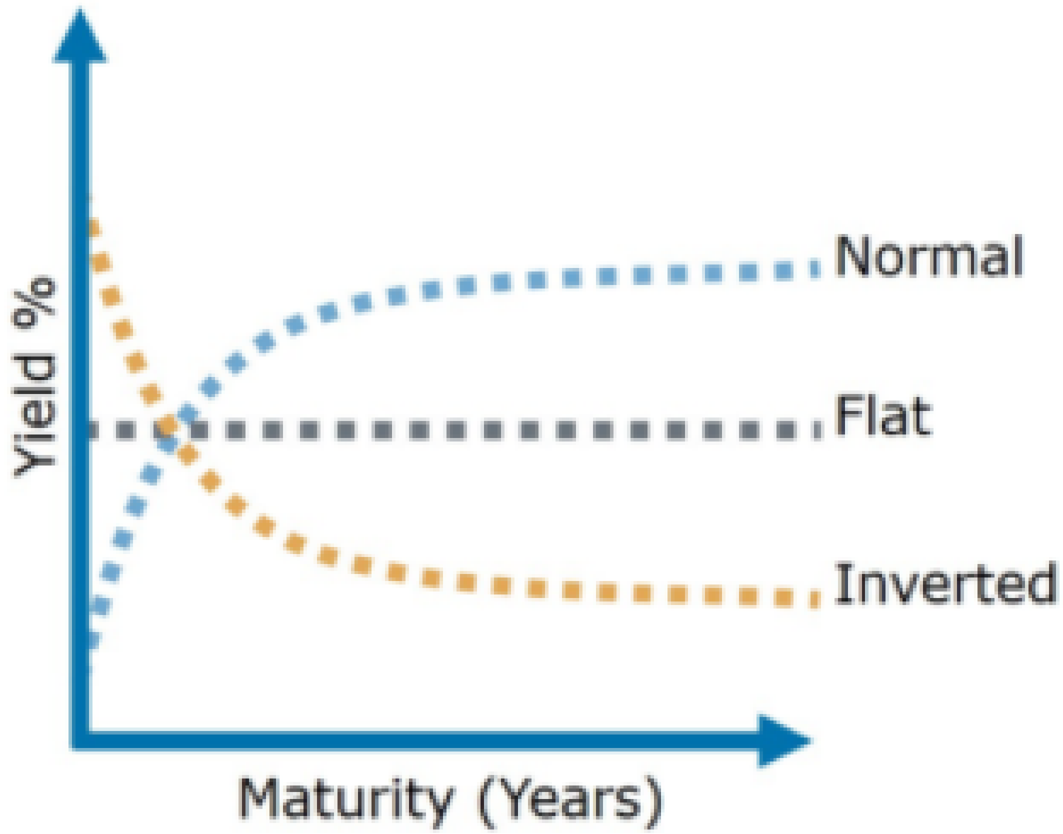
- Interpolation
- Jacobian
- Pricing & Risk

Detailed Notes

https://ssrn.com/abstract=3479833

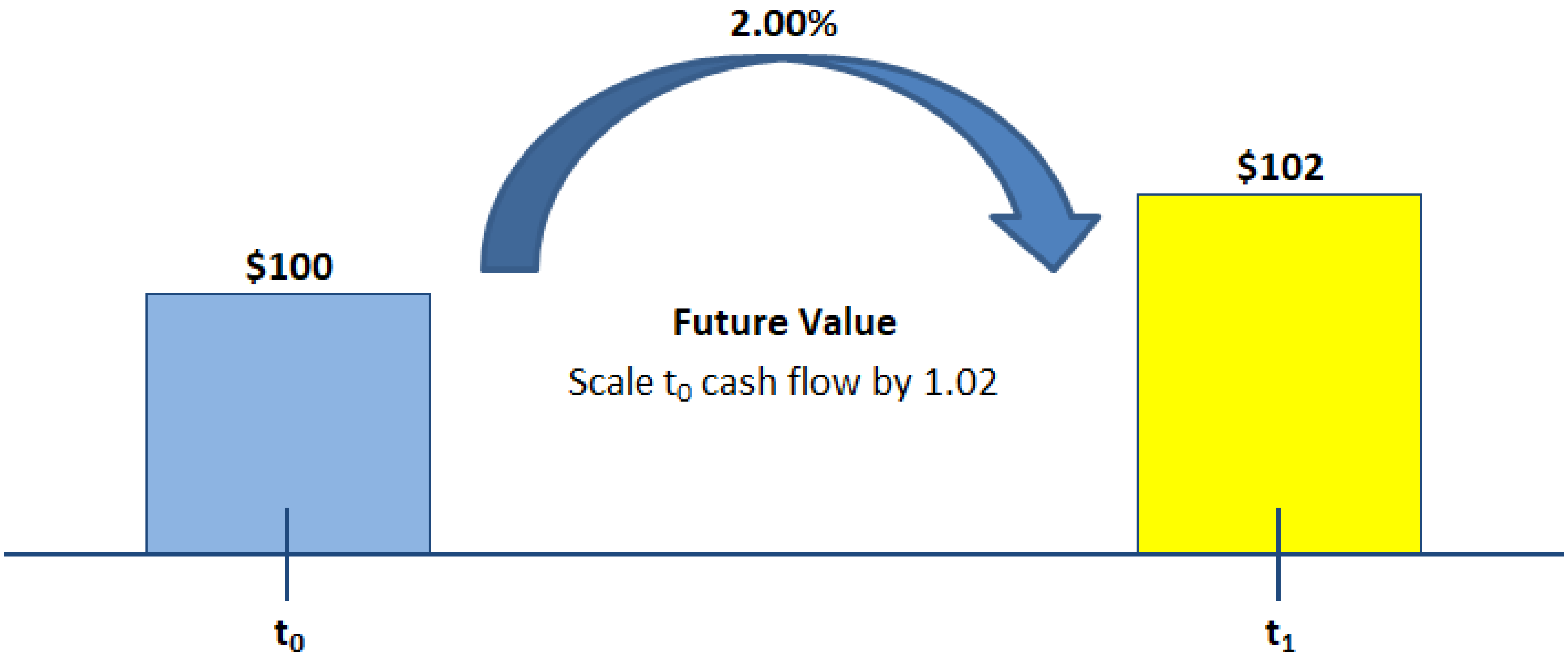
What is a Yield Curve?

- A curve of forward rates and discount factors over time
- Implied from liquid market instruments



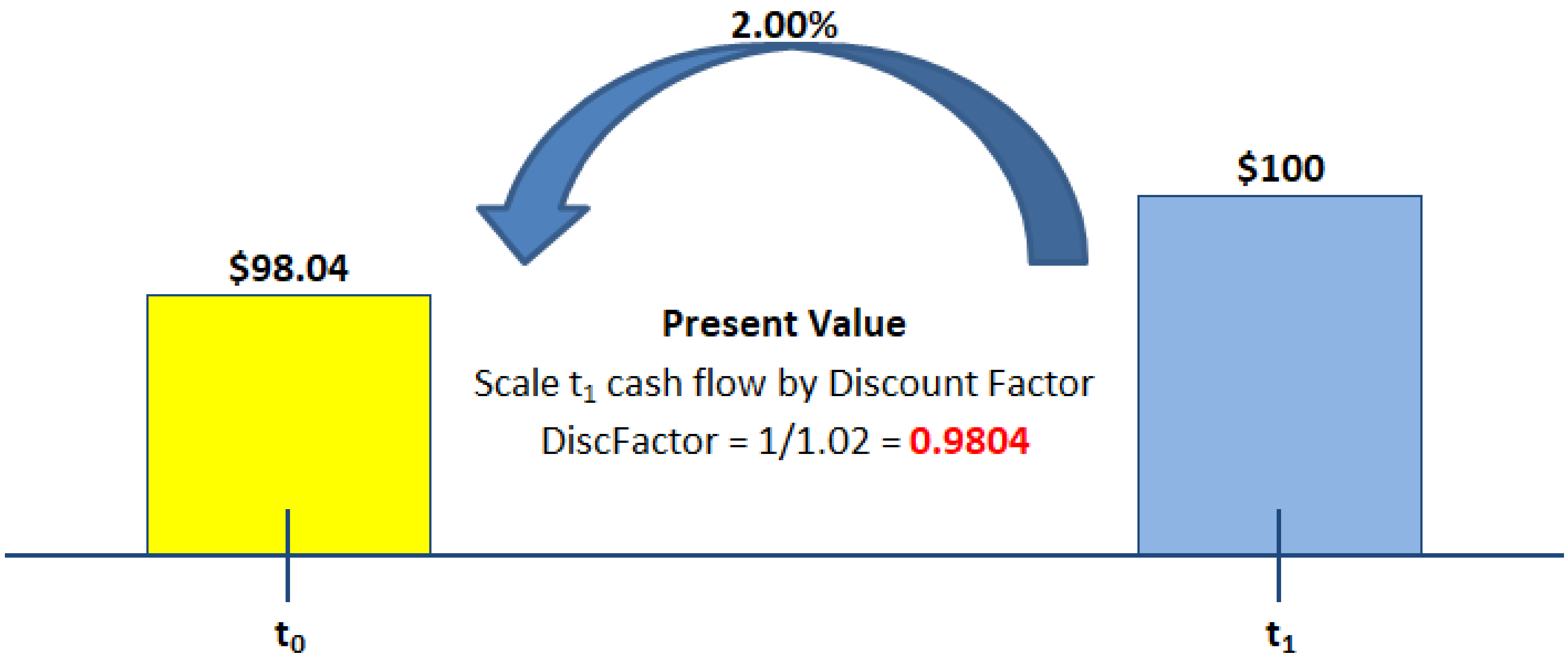
Time Value of Money

- Cash Deposits Earn Interest
- Future Value of Cash Includes Interest
- What is today's value of future cashflows?



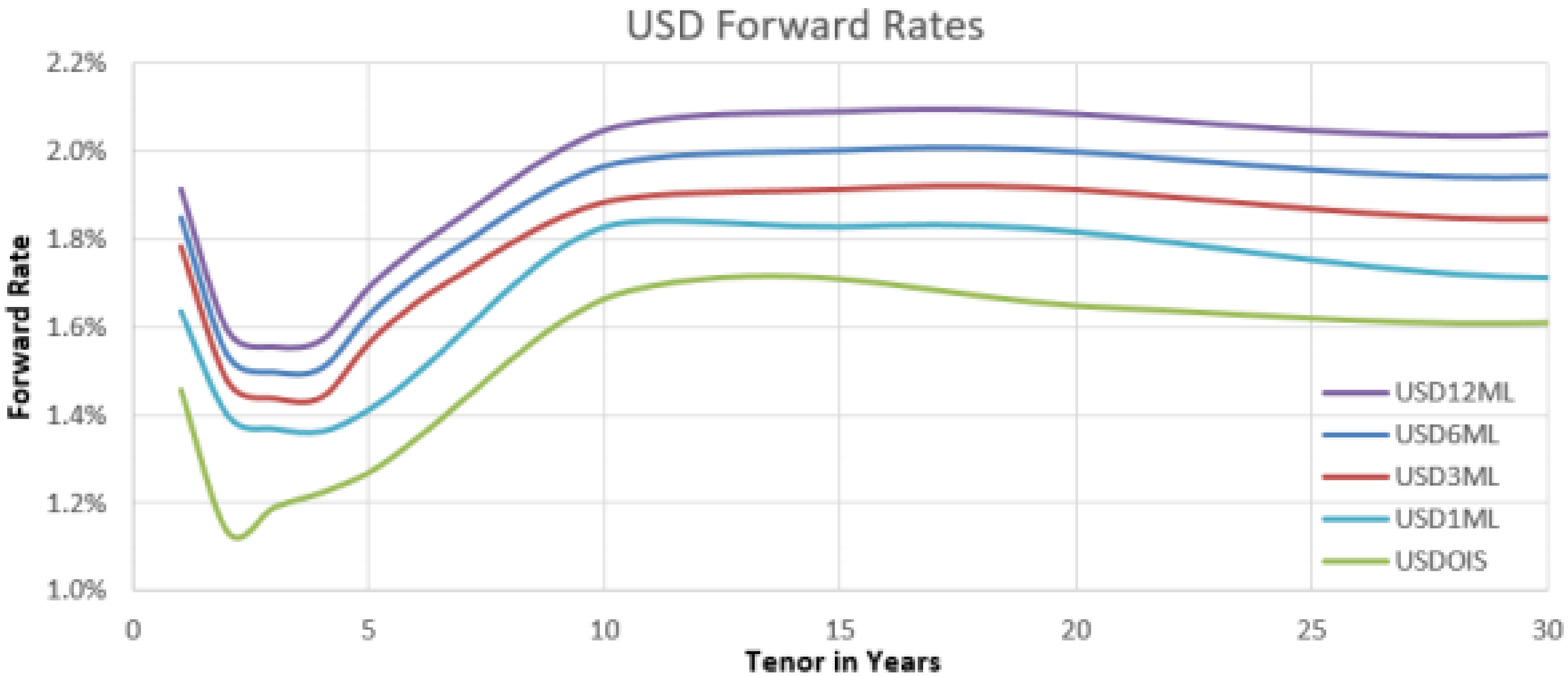
Discount Factors

- Time Value of Money
- Value of Cash Today
- Derived from Risk-Free Curve, Risk-Free?
- O/N Lending, Cleared, Margins, Collateralization



Forward Rates

- Libor based
- Includes a Credit Spread
- Borrowing over longer period increases risk



Types of Yield Curve

Curve Types

- Bond Yield Curves
- Swap Curves
- FX Forward Curves

Swap Curve Types

- Libor Curves
- OIS Curves
- Xccy, FX Forward & CSA Curves
- RFR Curves using ARR

Discounting with Collateral

USD CSA Discount Factors

- Implied from MtM Xccy Swap Spreads
- Typically Xccy trades have a USD leg and post USD collateral
- See https://ssrn.com/abstract=3278907

Non-USD CSA Discount Factors

- Implied from FX Forward Invariance (Replication Argument)
- See https://ssrn.com/abstract=3009281

Calibration Instruments

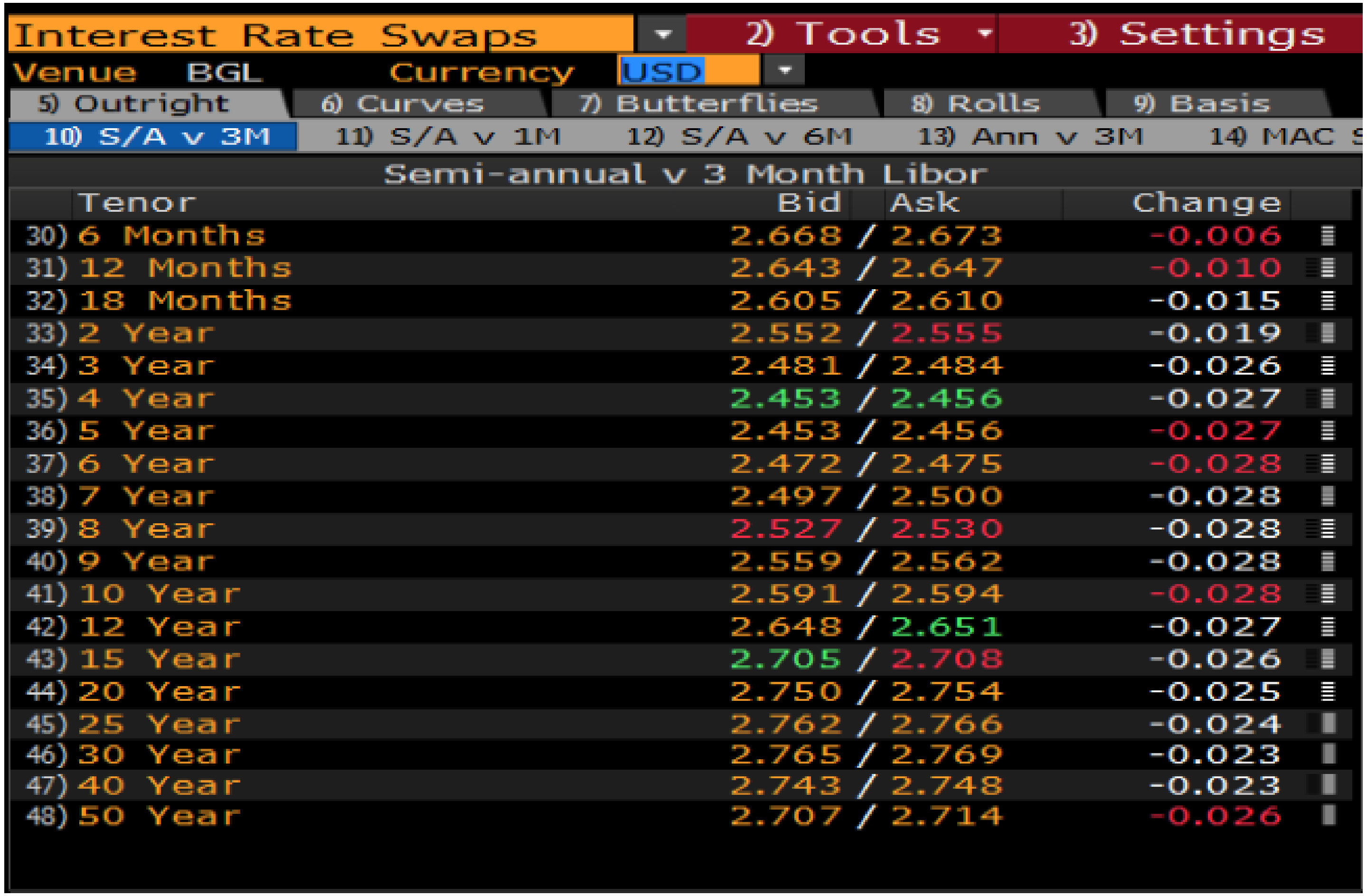
Instruments

- Cash Deposits
- FRAs 3M and 6M
- Futures 1M and 3M (IMM, Convexity Adj)
- Swaps: OIS, Libor, RFR
- Tenor Basis: LOB, LAB, AOB, LLB
- Xccy Basis: USD CSA
- FX Forwards: Non-USD CSA

Bloomberg Trading Venue

Bloomberg Trading Portal, BBTI

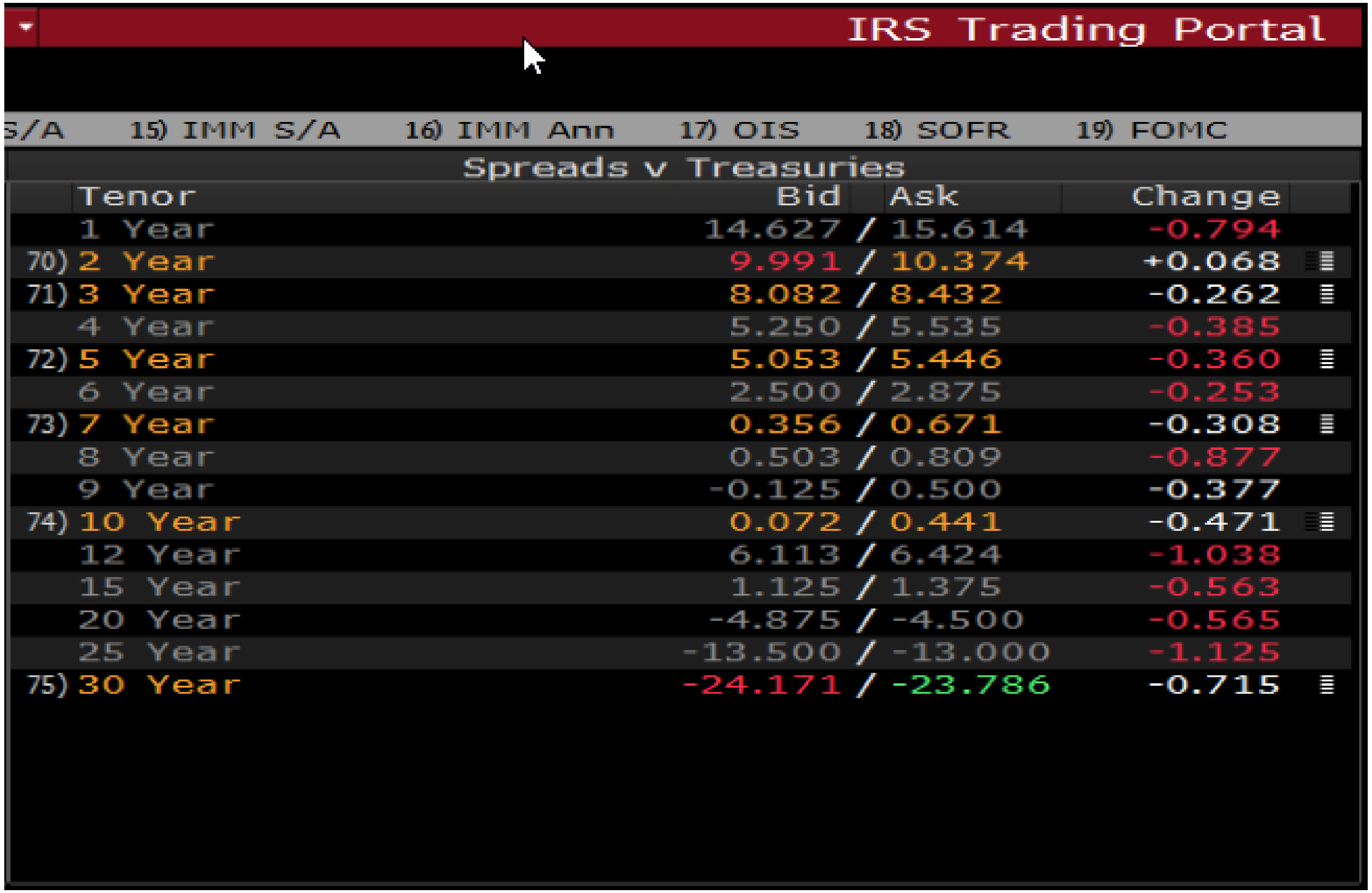
IRS Quote Pricing Precision: 1/10th Bps



Tenor	Bid	Ask	Change
30) 6 Months	2.668	2.673	-0.006
31) 12 Months	2.643	2.647	-0.010
32) 18 Months	2.605	2.610	-0.015
33) 2 Year	2.552	2.555	-0.019
34) 3 Year	2.481	2.484	-0.026
35) 4 Year	2.453	2.456	-0.027
36) 5 Year	2.453	2.456	-0.027
37) 6 Year	2.472	2.475	-0.028
38) 7 Year	2.497	2.500	-0.028
39) 8 Year	2.527	2.530	-0.028
40) 9 Year	2.559	2.562	-0.028
41) 10 Year	2.591	2.594	-0.028
42) 12 Year	2.648	2.651	-0.027
43) 15 Year	2.705	2.708	-0.026
44) 20 Year	2.750	2.754	-0.025
45) 25 Year	2.762	2.766	-0.024
46) 30 Year	2.765	2.769	-0.023
47) 40 Year	2.743	2.748	-0.023
48) 50 Year	2.707	2.714	-0.026

Swaps as a Spread Over US Treasuries

Par Rate = US Treasury Yield + Spread (Bps)



Tenor	Bid	Ask	Change
1 Year	14.627	15.614	-0.794
70) 2 Year	9.991	10.374	+0.068
71) 3 Year	8.082	8.432	-0.262
4 Year	5.250	5.535	-0.385
72) 5 Year	5.053	5.446	-0.360
6 Year	2.500	2.875	-0.253
73) 7 Year	0.356	0.671	-0.308
8 Year	0.503	0.809	-0.877
9 Year	-0.125	0.500	-0.377
74) 10 Year	0.072	0.441	-0.471
12 Year	6.113	6.424	-1.038
15 Year	1.125	1.375	-0.563
20 Year	-4.875	-4.500	-0.565
25 Year	-13.500	-13.000	-1.125
75) 30 Year	-24.171	-23.786	-0.715

Interpolation

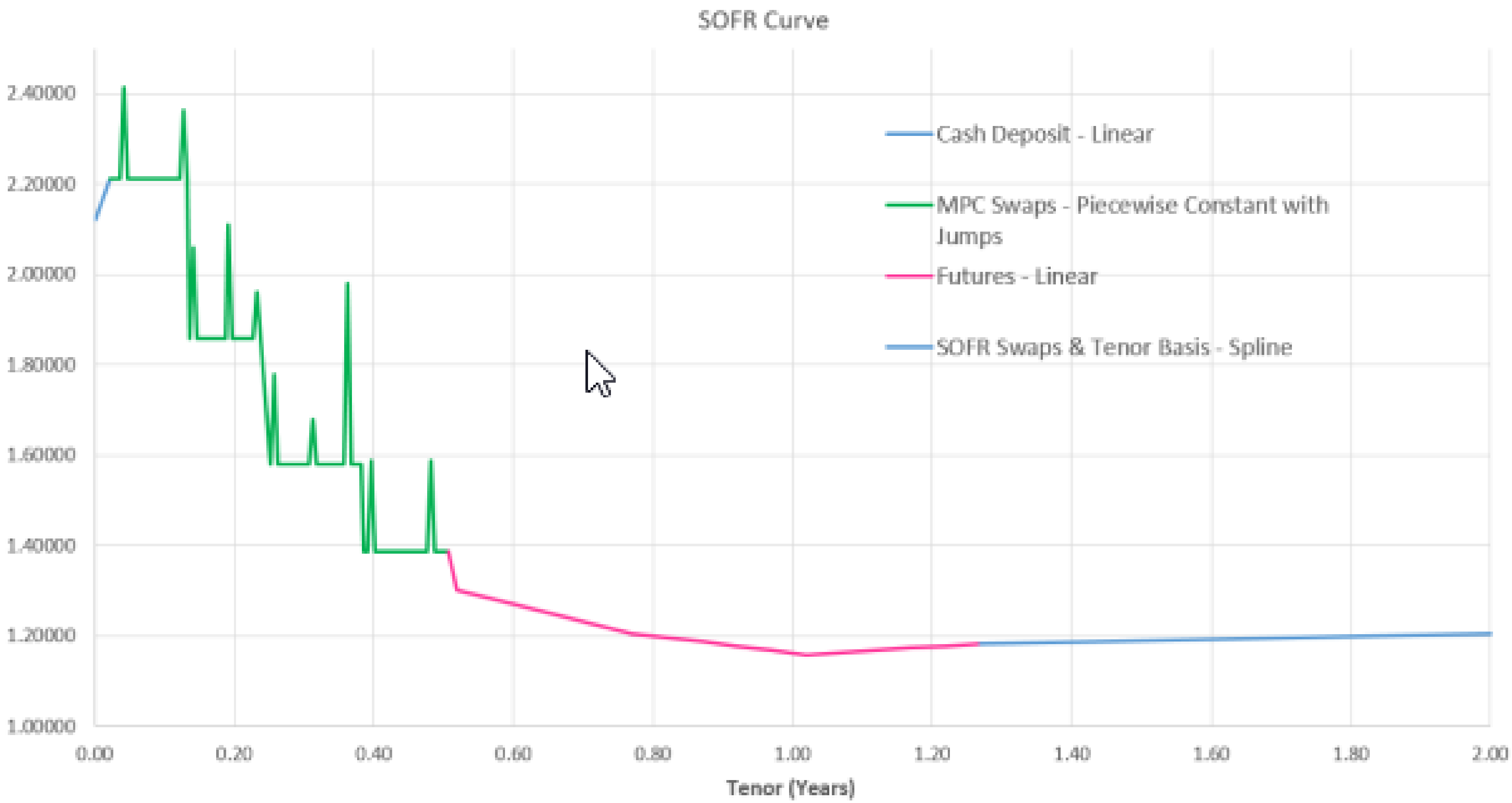
- Intrinsically part of the curve framework
- Interpolate on Forwards or Discount Factors?
- Local vs Global Interpolation & Implications for Risk

Instrument Behaviour

- Piecewise Constant: Central Bank Swaps (MPC/FOMC)
- Jumps & Turns: Policy Meeting Dates, Year End-Squeezes
- Linear: Futures
- Smooth Spline: Swaps

Curve Shape

Hybrid / Mixed Interpolation:
Linear, Constant, Jumps, Linear, Smooth Spline

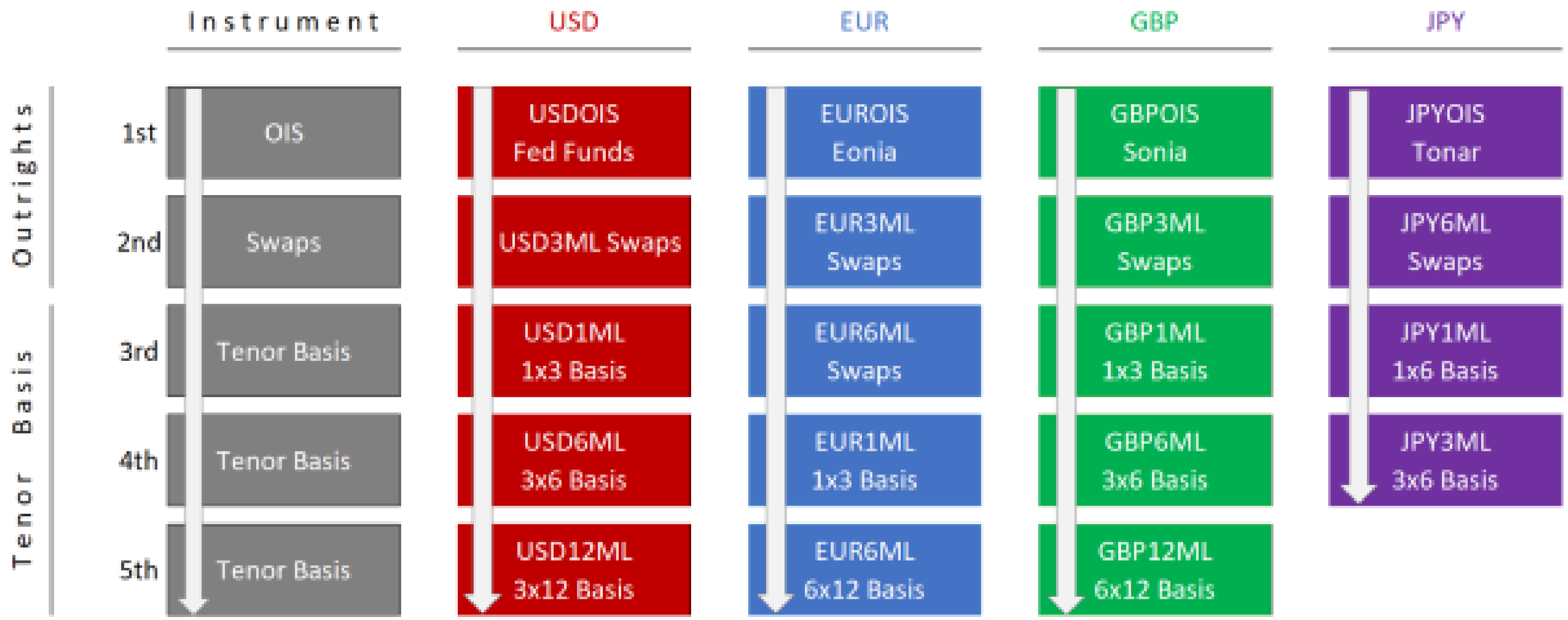


Calibrate Curves Independently

- Basis Instruments e.g. LOB circular dependency
- Risk can have Ghost Instruments?
- Complex Build Order - Complicates real-time curves

Single Curve Dependencies

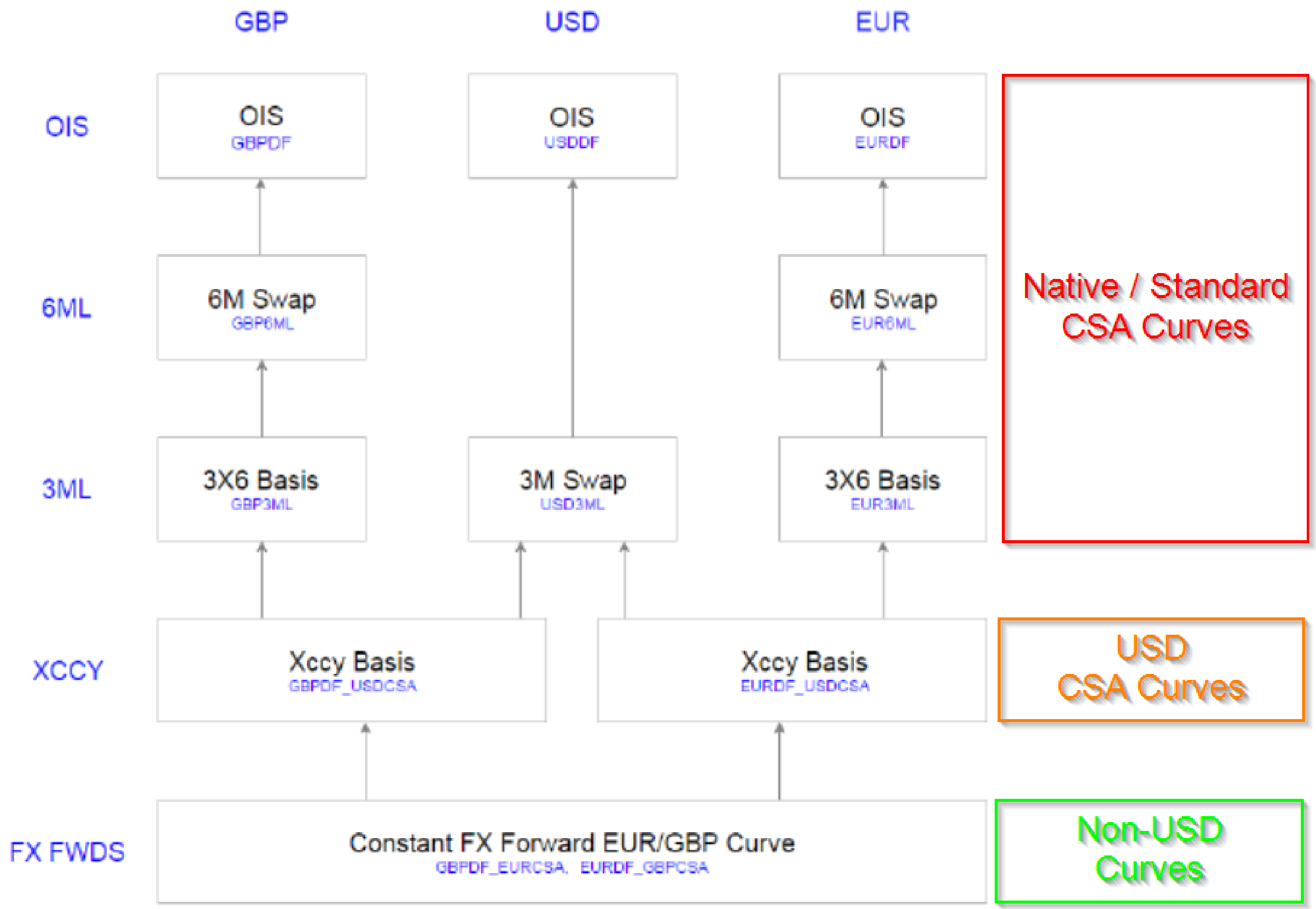
Build Outright then Basis Curves



Single Curves Multi-Ccy Dependency Tree

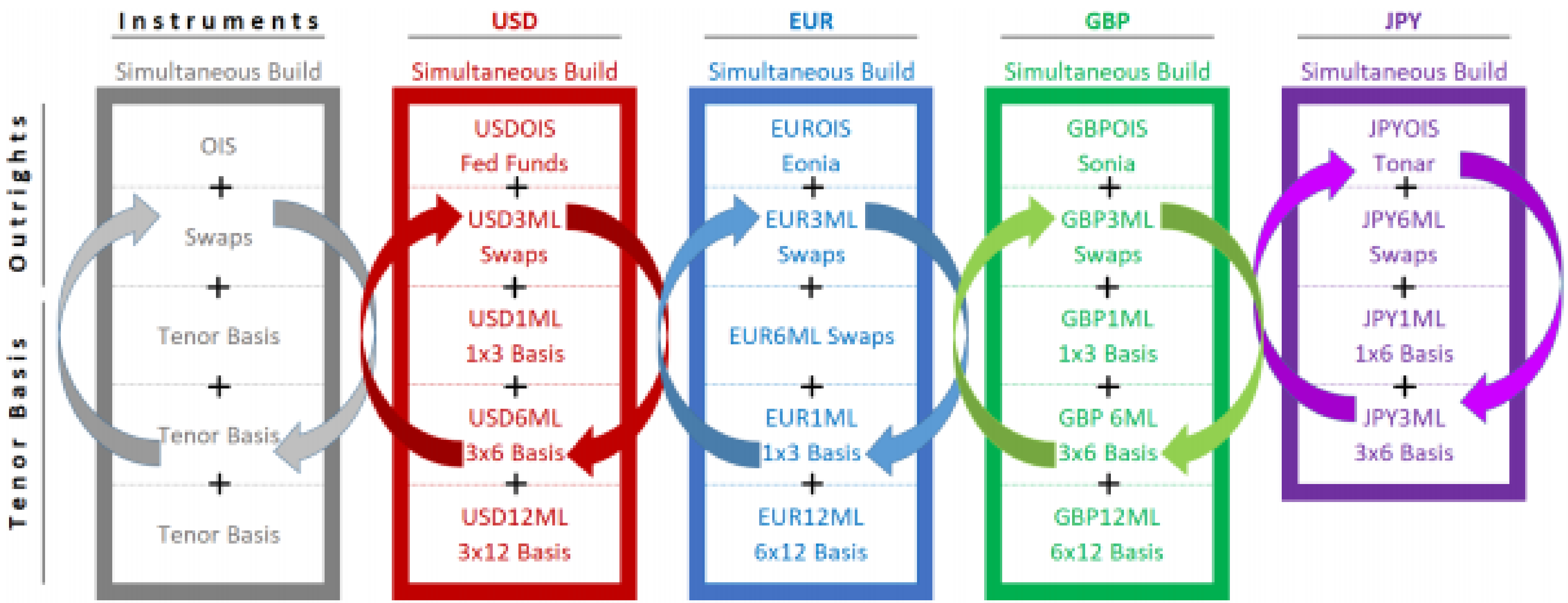
EURDF with GBP Collateral

Build Native CSA, USD CSA then Non-USD CSA



Calibrate Curves Simultaneously

- Price All Instruments Simultaneously
- Solve for All Forwards & Discount Factors Simultaneously
- More Accurate for Risk Calculations (No Ghost Instruments?)



Calibration

Calibration Steps

1. Select State Variable - Ideally Fwds (DF is bad - why?)
2. Select Functional Form and/or Interpolation Scheme
3. Solve or Minimize

Potential Issues

- Speed, Accuracy, Risk & Stability
- Matrix Size and Invertibility Issues
- Difficult to perfect the curve shape
- Bootstrapping vs Global Optimization
  Can we bootstrap a Spline?

Advanced Features

- Ticking Curves, Auto-Execution & Auto-Hedging
- Requires Jacobian for Fast Rebuilds & Analytical Risk
- Jumps, Overlay Curves & Turn-of-Year Effects (ToY)
- Advanced Hybrid/Mixed Interpolation Schemes
- CTD Curves using Collateral Switch Options
- Machine Learning Classifiers
  e.g. PCA Analysis, SVM (Rich/Cheap)

Multi-Dimensional Solvers & Optimization

- Examples: Gradient-Decent, Newton-Raphson, Secant, ...
- Gradient Decent Solvers Calculate Slope / First Derivative
- Keep Jacobian for Quick Rebuilds & Analytical Risk

Newton-Raphson

$$X_{n+1} = X_n - \frac{f(X_n)}{f'(X_n)}$$

or equivalently

$$X_{n+1} = X_n - \mathcal{J}^{-1} f(X_n)$$

where $\mathcal{J}$ is the Jacobian

Curve Jacobian

- First Order Derivatives
- Numerical vs Analytical
- Useful for Curve Updates & Analytical Risk

Jacobian, dParRate/dLibor (dp/dL)

p = PV(Float Leg)/Annuity(Fixed Leg)

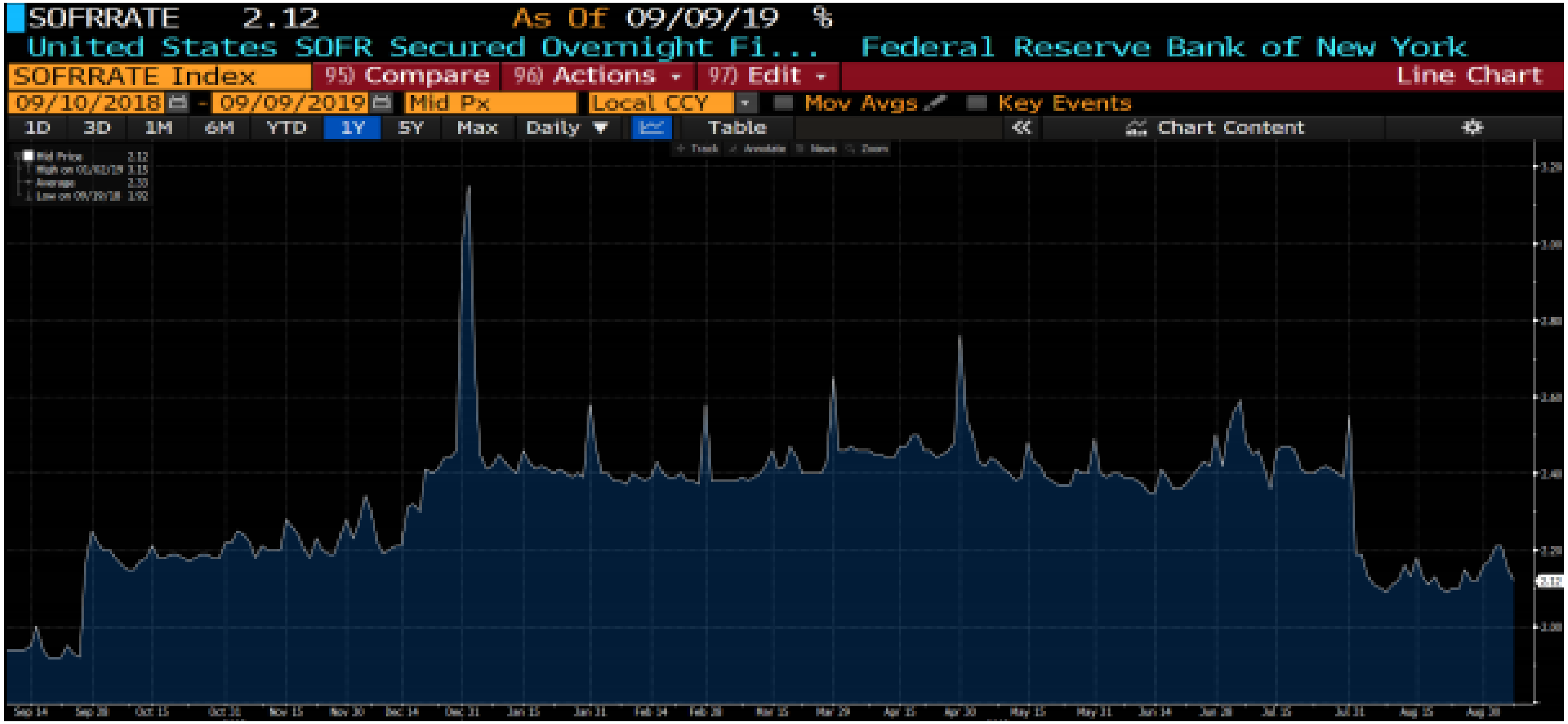
$dp_i/dL_j = N\tau_j DF_j \,/\, A_i(\text{Fixed}) = DF_j \,/\, A_i(\text{Fixed})$, since N=1 and τ=1

$$J = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial x_1} & \cdots & \frac{\partial F_m}{\partial x_n} \end{bmatrix}$$

	dL_{1Y}^{OIS}	dL_{2Y}^{OIS}	dL_{3Y}^{OIS}	dL_{4Y}^{OIS}	dL_{5Y}^{OIS}	dL_{1Y}^{IRS}	dL_{2Y}^{IRS}	dL_{3Y}^{IRS}	dL_{4Y}^{IRS}	dL_{5Y}^{IRS}
dP_{1Y}^{OIS}	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
dP_{2Y}^{OIS}	0.50	0.50	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
dP_{3Y}^{OIS}	0.34	0.33	0.33	0.00	0.00	0.00	0.00	0.00	0.00	0.00
dP_{4Y}^{OIS}	0.25	0.25	0.25	0.25	0.00	0.00	0.00	0.00	0.00	0.00
dP_{5Y}^{OIS}	0.21	0.20	0.20	0.20	0.19	0.00	0.00	0.00	0.00	0.00
dP_{1Y}^{IRS}	0.00	0.00	0.00	0.00	0.00	1.00	0.00	0.00	0.00	0.00
dP_{2Y}^{IRS}	0.00	0.00	0.00	0.00	0.00	0.50	0.50	0.00	0.00	0.00
dP_{3Y}^{IRS}	0.00	0.00	0.00	0.00	0.00	0.34	0.33	0.33	0.00	0.00
dP_{4Y}^{IRS}	0.00	0.00	0.00	0.00	0.00	0.25	0.25	0.25	0.25	0.00
dP_{5Y}^{IRS}	0.00	0.00	0.00	0.00	0.00	0.21	0.20	0.20	0.20	0.19

Jumps & Turn of Year (ToY)

- Meeting Dates & Liquidity Squeezes
- Year & Quarter End Fund Rebalancing



Overlay Curve

$$f^*(t, T) = f(t, T) + \epsilon.1_{T_S \leq T \leq T_E}$$

The trader models and specifies a table of jumps a-priori.

If the forward fixing date T is within the jump range $[T_S, T_E]$ then the adjusted forward rate f^* is the unadjusted forward f plus the pre-specified jump ϵ.

So what is wrong with Libor?

Libor Problem

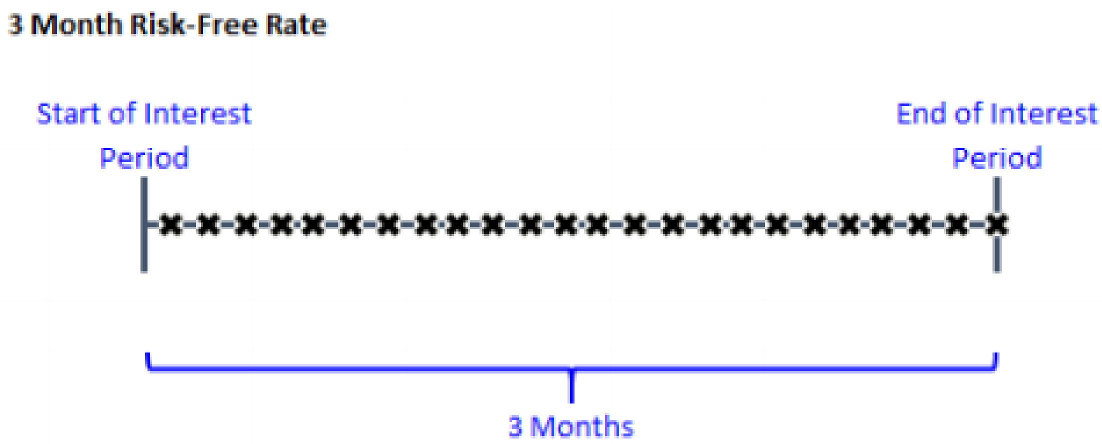
So what is wrong with Libor?

- Libor is used to reference over USD 200 trillion of financial contracts
- It has become illiquid and no longer representative of actual borrowing levels
- The rate is determined by a small number of transactions in a handful of geographies
- Can be subject to **'expert'** panel judgement

Alternative Reference Rates, ARRs

LIBOR: Forward looking term rate set in advance

ARR: Backward looking compounded rate set in arrears



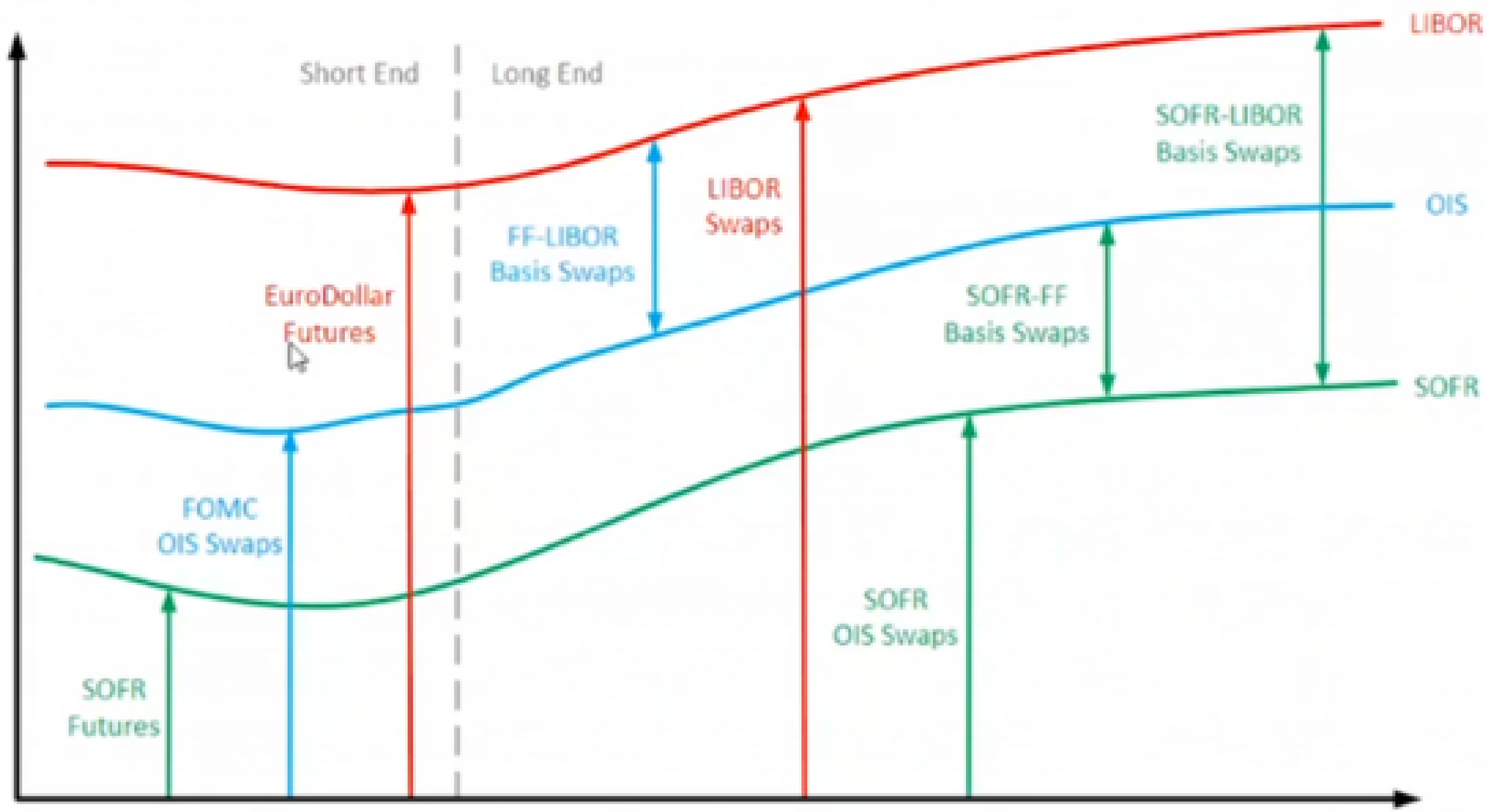
What does this mean for European Swaptions? To become Asian?

New Instruments

SOFR Curve

Instrument	Term (Years)	Quote	Interpolation Style
Cash Deposit	0.00	2.12000	Linear
Monetary Policy SOFR Swap	0.02	2.21266	Piecewise-Constant with Jumps
Monetary Policy SOFR Swap	0.14	1.85987	Piecewise-Constant with Jumps
Monetary Policy SOFR Swap	0.25	1.57939	Piecewise-Constant with Jumps
Monetary Policy SOFR Swap	0.39	1.38860	Piecewise-Constant with Jumps
Future 5	0.52	98.69748	Linear
Future 6	0.77	98.79385	Linear
Future 7	1.02	98.84050	Linear
Future 8	1.27	98.81677	Linear
SOFR Swap	3	1.22559	Spline
SOFR Swap	5	1.20502	Spline
SOFR Swap	7	1.23028	Spline
SOFR-OIS Basis Swap	10	0.01000	Spline
SOFR-OIS Basis Swap	15	0.02500	Spline
SOFR-OIS Basis Swap	20	0.05000	Spline
SOFR-LIBOR Basis Swap	30	0.07500	Spline
SOFR-LIBOR Basis Swap	40	0.08000	Spline
SOFR-LIBOR Basis Swap	50	0.10000	Spline

Arbitrage Opportunities?

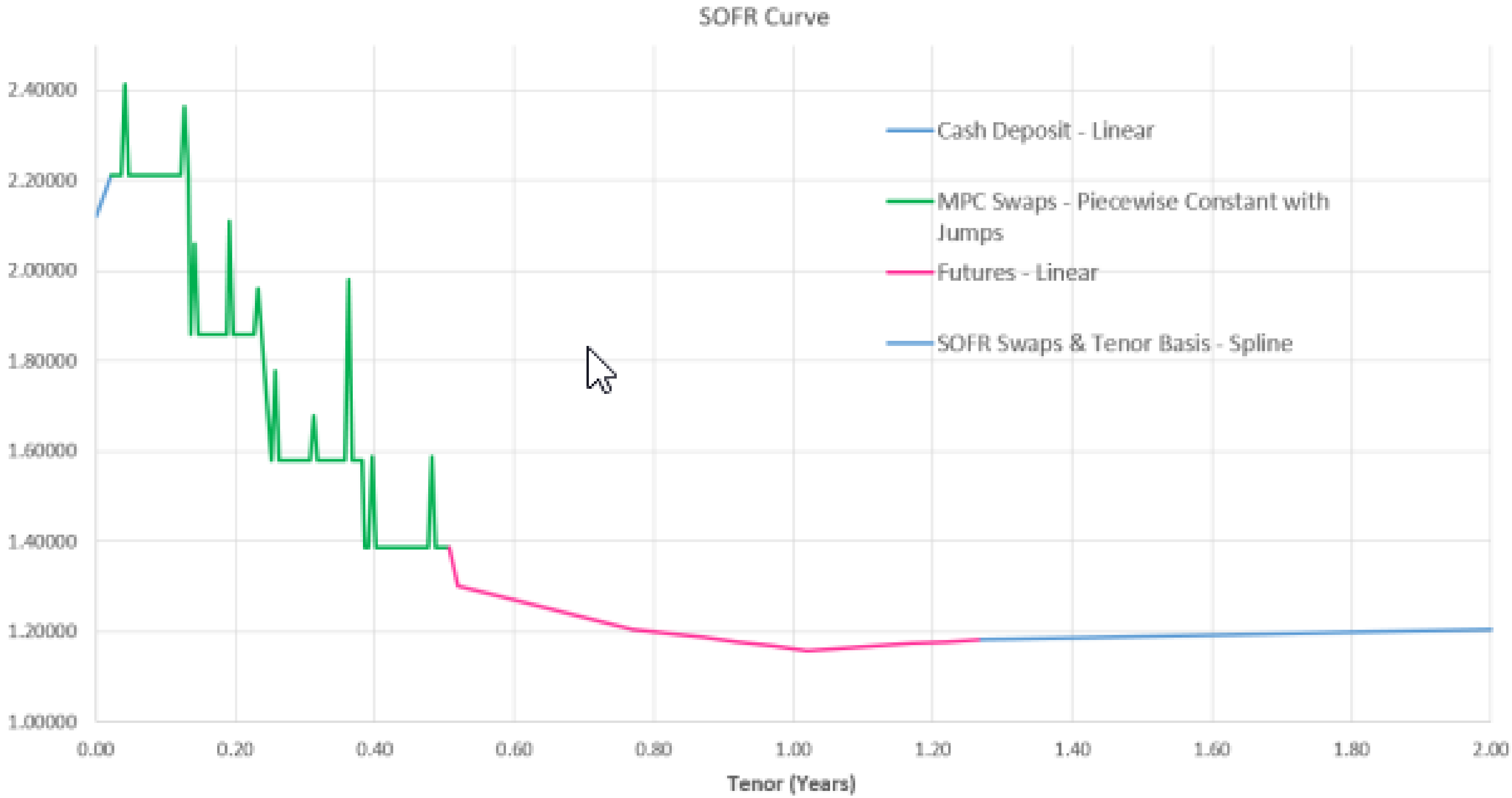


RFR Curves using ARRs

Key Differences

- Backward vs Forward Looking Rates
- Replacement Term Rates?
- Futures Roll Date Changes Advance vs Arrears
- ARR Curves Require Fixing Tables
- Stub Rate Calculations Differences
- Convexity Adjustment Methodology Differences
- Legal issues & disputes with IBOR fallbacks
- Some complex transactions have no IBOR fallback

RFR Curve Shape



Why Hard to Set-up?

Demanding Multi-Curve Requirements?

- Over 100 instruments must be calibrated simultaneously
- Must solve for 10,000 forecast rates and discount factors
- Must be able to price a wide variety of instruments
- Mixture of IBOR and ARR Curves, complicates Xccy set-up
- Combination of backward and forward looking interest rates
- Fixing tables and pro-rated future convexity adjustments
- Hybrid/Mixed Interpolation required with jumps and turns
- Instruments must reprice to 1/10th Bps i.e. 0.000001
- Speed of 5-10 milliseconds required for modest performance
- Risk sensitivities also required

Let’s work through an example

Curve Calibration Example

$$X_{n+1} = X_n - \mathcal{J}^{-1} f(X_n)$$

Multi-Dimensional Newton-Raphson Algorithm

$X_{n+1} = X_n - J^{-1} f(X_n)$

Tolerance	RMSE
1.00E-08	8.72E-12

Iteration: 4

Curve	Term	Time, t	X_{n+1}	X_n	Initial Guess X_0	$f(X_n)$	Epsilon
USDOIS	1Y	1.00	**1.43591%**	1.43591%	2.00000%	0.00000%	0.00E+00
USDOIS	2Y	2.00	**1.23323%**	1.23323%	2.00000%	0.00000%	2.69E-12
USDOIS	3Y	3.00	**1.25107%**	1.25107%	2.00000%	0.00000%	3.86E-12
USDOIS	4Y	4.00	**1.29130%**	1.29130%	2.00000%	0.00000%	1.00E-12
USDOIS	5Y	5.00	**1.39782%**	1.39782%	2.00000%	0.00000%	-3.89E-12
USD3ML	1Y	1.00	**1.70896%**	1.70896%	2.00000%	0.00000%	0.00E+00
USD3ML	2Y	2.00	**1.47359%**	1.47359%	2.00000%	0.00000%	3.13E-12
USD3ML	3Y	3.00	**1.49531%**	1.49531%	2.00000%	0.00000%	4.44E-12
USD3ML	4Y	4.00	**1.55934%**	1.55934%	2.00000%	0.00000%	5.28E-14
USD3ML	5Y	5.00	**1.62999%**	1.62999%	2.00000%	0.00000%	-2.89E-12

USDOIS Discount Factors

Integrate USDOIS Forward Polynomial

Time, t	DiscFactor	Integrand
1.00	**0.982281**	1.78781%
2.00	**0.969579**	3.08936%
3.00	**0.957671**	4.32509%
4.00	**0.945574**	5.59628%
5.00	**0.933074**	6.92710%

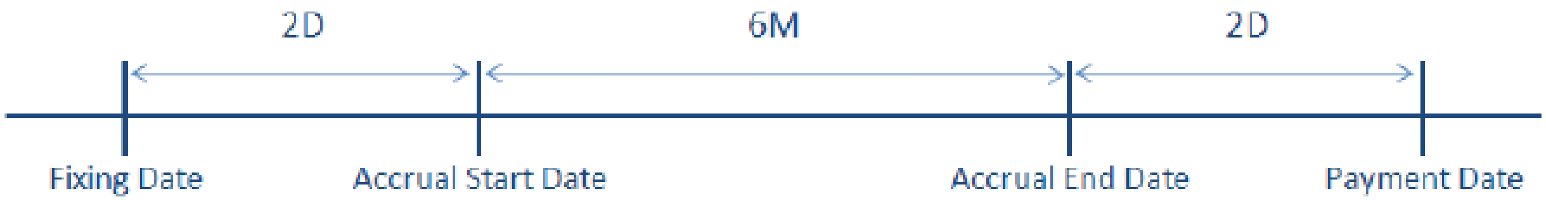
Update Solver

Interest Rate Swap Pricing

Swap Specification & Pricing

To specify a swap many parameters are required to generate the swap cashflow schedules accurately. To price a swap we require Libor forecast rates, OIS discount rates and a Swap pricing formula.

$$PV^{Swap} = N \sum_{\forall i} r^{Fixed} \tau_i P(t_0, t_i) - N \sum_{\forall j} (L_j + s)\, \tau_j P(t_0, t_j)$$



IRS Pricing Example

USD 1MM 5Y IRS Pay Fixed @ 1.0%

Swap Trade Details

Payer/Receiver	PAYER
Currency	USD
Notional, N	1,000,000
Fixed Rate, r^{Fixed}	1.0000%
Fixed Frequency	ANNUAL
Float Frequency	ANNUAL
Libor Spread, s	0.00
Tenor, T	5.00

Fixed Leg

Row	Accrual Start	Accrual End	Pay Date	t_i	N	r^{Fixed}	τ_i	$P(t_E, t_i)$	PV^{Fixed}
1	05-Apr-21	05-Apr-22	05-Apr-22	1.00	1,000,000	1.0000%	1.00	0.982281	9,823
2	05-Apr-22	05-Apr-23	05-Apr-23	2.00	1,000,000	1.0000%	1.00	0.969579	9,696
3	05-Apr-23	04-Apr-24	04-Apr-24	3.00	1,000,000	1.0000%	1.00	0.957671	9,577
4	04-Apr-24	04-Apr-25	04-Apr-25	4.00	1,000,000	1.0000%	1.00	0.945574	9,456
5	04-Apr-25	04-Apr-26	04-Apr-26	5.00	1,000,000	1.0000%	1.00	0.933074	9,331
6									
7									

Float Leg

Row	Fixing Date	Accrual Start	Accrual End	Pay Date	t_j	N	l_{j-1}	s	$l_{j-1} + s$	τ_j	$P(t_E, tj)$	PV^{Float}
1	05-Apr-21	05-Apr-21	05-Apr-22	05-Apr-22	1.00	1,000,000	1.7090%	0.00	1.7090%	1.00	0.982281	16,787
2	05-Apr-22	05-Apr-22	05-Apr-23	05-Apr-23	2.00	1,000,000	1.4736%	0.00	1.4736%	1.00	0.969579	14,288
3	05-Apr-23	05-Apr-23	04-Apr-24	04-Apr-24	3.00	1,000,000	1.4953%	0.00	1.4953%	1.00	0.957671	14,320
4	04-Apr-24	04-Apr-24	04-Apr-25	04-Apr-25	4.00	1,000,000	1.5593%	0.00	1.5593%	1.00	0.945574	14,745
5	04-Apr-25	04-Apr-25	04-Apr-26	04-Apr-26	5.00	1,000,000	1.6300%	0.00	1.6300%	1.00	0.933074	15,209
6												
7												
8												
9												
10												

Swap Pricing

Swap PV	27,466
Fixed Leg PV	-47,882
Float Leg PV	75,348
Par Rate	1.57363%

Swap Risk

PV01	-479
Numerical DV01	-471
Analytical DV01	-471
+/-	0

Useful IRS Pricing Formulae

Fixed Leg

$$PV(Fixed) = N \times r^{Fixed} \underbrace{\sum_{\forall i} \tau_i P(t_0, t_i)}_{Annuity}$$

Float Leg

$$PV(Float) = N \sum_{\forall j} (L_j + s) \tau_j P(t_0, t_j)$$

Swap Price

$$PV(Swap) = \phi (PV(Fixed) - PV(Float))$$

Swap Rate

$$ParRate = \frac{PV(Float)}{N \times Annuity}$$

IRS Analytical Risk

Swap Delta, dS/dP = dS/dL . dL/dP . Shift Size

Curve Jacobian, J = dL/dP
Change in Libor rate per unit change in market par rates

	dP_{1Y}^{IRS}	dP_{2Y}^{IRS}	dP_{3Y}^{IRS}	dP_{4Y}^{IRS}	dP_{5Y}^{IRS}
dL_{1Y}^{IRS}	1.00	0.00	0.00	0.00	0.00
dL_{2Y}^{IRS}	-1.01	2.01	0.00	0.00	0.00
dL_{3Y}^{IRS}	0.00	-2.04	3.04	0.00	0.00
dL_{4Y}^{IRS}	0.00	0.00	-3.08	4.08	0.00
dL_{5Y}^{IRS}	0.00	0.00	0.00	-4.13	5.13

Shift Size, dP
Change in market par rates

	Shift, Bps	Shift, %
dP_{1Y}^{IRS}	1.00	0.01%
dP_{2Y}^{IRS}	1.00	0.01%
dP_{3Y}^{IRS}	1.00	0.01%
dP_{4Y}^{IRS}	1.00	0.01%
dP_{5Y}^{IRS}	1.00	0.01%

Swap Jacobian, dS/dL
Change in swap value per unit change in Libor Rate

	dL_{1Y}	dL_{2Y}	dL_{3Y}	dL_{4Y}	dL_{5Y}	
dS_{1Y}^{IRS}	982,281	0	0	0	0	
dS_{2Y}^{IRS}	982,281	969,579	0	0	0	
dS_{3Y}^{IRS}	982,281	969,579	957,671	0	0	
dS_{4Y}^{IRS}	982,281	969,579	957,671	945,574	0	
dS_{5Y}^{IRS}	982,281	969,579	957,671	945,574	933,074	
$dS_{4Y,5Y}^{IRS}$	0	0	0	0	933,074	*
$dS_{4.5Y}^{IRS}$	982,281	969,579	957,671	945,574	466,537	**

Risk, dS/dP = dS/dL x dL/dP
Change in swap value per unit change in market par rates

	dP_{1Y}^{IRS}	dP_{2Y}^{IRS}	dP_{3Y}^{IRS}	dP_{4Y}^{IRS}	dP_{5Y}^{IRS}	Total	
dS_{1Y}^{IRS}	98	0	0	0	0	98	IRS(1Y)
dS_{2Y}^{IRS}	0	195	0	0	0	195	IRS(2Y)
dS_{3Y}^{IRS}	0	0	291	0	0	291	IRS(3Y)
dS_{4Y}^{IRS}	0	0	0	386	0	386	IRS(4Y)
dS_{5Y}^{IRS}	0	0	0	0	479	479	IRS(5Y)
$dS_{4Y,5Y}^{IRS}$	0	0	0	-386	479	93	Forward IRS(4Y,5Y)
$dS_{4.5Y}^{IRS}$	0	0	0	193	239	432	IRS(4.5Y)

$$\text{Swap Delta} = \frac{dS}{dP} = \frac{dS}{dL}\cdot\frac{dL}{dP}\times \text{Shift Size}$$

* Forward Starting Swap: Start 4Y, End 5Y, Equivalent to Long 5Y + Short 4Y
** 4.5Y IRS Carries 50% Risk of 4Y and 50% Risk of 5Y IRS

Summary

Yield Curves

- Yield curves calculate forward rates & discount factors
- There are different types of curves
- Calibration instruments have unique behaviour

Calibration

- Interpolation is key part of calibration
- Jacobian is useful for fast curve updates & analytical risk
- Libor rates are being replaced with ARRs
- We provided an example of pricing & risk

Detailed Notes

https://ssrn.com/abstract=3479833

Thank You!

References

1. Yield Curve Construction & Libor Reform
https://ssrn.com/abstract=3479833

2. Collateralization & CSA Fundamentals
https://ssrn.com/abstract=3035648

3. Discounting with Collateral
https://ssrn.com/abstract=3009281

4. An Interest Rate Swap Primer
https://ssrn.com/abstract=2815495

5. Interest Rate Modeling: Volume I-III
Atlantic Financial Press - Vladimir Piterbarg

6. Interest Rate Models - Theory & Practice
Springer - Damiano Brigo, Fabio Mercurio